Weyl's Gauge Principle

The extension of geometry Weyl used to place electromagnetism on a geometrical basis allowed him to use his gauge principle to derive the Maxwell equations of electromagnetism. We shall now present this derivation.

Weyl defined the gauge potentials as

$$\phi_i \equiv \frac{\partial \ln f^{\frac{1}{2}}}{\partial x^i}. \tag{1}$$

Now the electromagnetic field tensor is given by

$$F_{ij} \equiv \phi_{i,j} - \phi_{j,i}. \tag{2}$$

The field tensor given by Equation (2) has 16 components when the indices range over four dimensions. We would like to determine the field equations for these components. The quickest, though not the only, way is to consider the four dimensions to be x^0=ict, x^1=x, x^2=y and x^3=z. The field tensor is then defined to be

$$F_{ij} = \begin{vmatrix} 0 & iE_1 & iE_2 & iE_3 \\ -iE_1 & 0 & B_3 & -B_2 \\ -iE_2 & -B_3 & 0 & B_1 \\ -iE_3 & B_2 & -B_1 & 0 \end{vmatrix}. \tag{3}$$

Using Bianchi's identities

$$\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0$$

and the various combinations of the indices 0, 1, 2, 3 we obtain the field equations

$$\begin{gathered} \overline{\nabla} \bullet \overline{B} = 0 \\ \overline{\nabla} \times \overline{E} + \frac{1}{c}\frac{\partial \overline{B}}{\partial t} = 0. \end{gathered} \tag{4}$$

The definition of the four-vector current density

$$\frac{\partial F_{ij}}{\partial x^i} \equiv \frac{4\pi}{c} J_i \tag{5}$$

yields the equations

$$\begin{gathered} \overline{\nabla} \bullet \overline{E} = 4\pi\rho \\ \overline{\nabla} \times \overline{B} - \frac{1}{c}\frac{\partial \overline{E}}{\partial t} = \frac{4\pi \overline{J}}{c}. \end{gathered} \tag{6}$$

In addition to these field equations there is the statement of conservation of charge where

$$\frac{\partial J_i}{\partial x^i} = 0, \quad i = 0,1,2,3,$$

so that

$$\frac{\partial \rho}{\partial t} + \overline{\nabla} \bullet \overline{J} = 0. \qquad (7)$$

For ease in future reference to these five field equations they may be rewritten as

$$\begin{aligned}
&\overline{\nabla} \bullet \overline{B} = 0 && [a] \\
&\frac{1}{c}\frac{\partial \overline{B}}{\partial t} + \overline{\nabla} \times \overline{E} = \overline{0} && [b] \\
&\overline{\nabla} \times \overline{B} - \frac{1}{c}\frac{\partial \overline{E}}{\partial t} = \frac{4\pi \overline{J}}{c} && [c] \\
&\overline{\nabla} \bullet \overline{E} = 4\pi\rho && [d] \\
&\frac{\partial \rho}{\partial t} + \overline{\nabla} \bullet \overline{J} = 0 && [e]
\end{aligned} \qquad (8)$$

The five field equations in Equations (8) are universally known as the Maxwell field equations and the above process shows how they may be derived from Weyl’s Gauge Principle. Weyl’s derivation of these equations from his gauge function was the origin of the term “gauge field equations” applied to Maxwell’s equations.